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NON-LINEAR DYNAMIC ANALYSIS OF ANISOTROPIC CYLINDRICAL SHELLS CONTAINING A FLOWING FLUID

AOUNI A. LAKIS and ANDRÉ LAVEAU
Department of Mechanical Engineering, Ecole Polytechnique de Montréal,
Campus de l'Université de Montréal, C.P. 6079, Succ. A, Montréal, Québec,
Canada H3C 3A7

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Abstract—An analytical model is presented to predict the influence of non-linearities associated with fluid flow on the dynamic behaviour of a structure consisting of shells and a surrounding fluid medium. The model requires the use of two linear operators to control the equilibrium of the shell and the velocity potential, a linear boundary condition of impermeability and a non-linear dynamic boundary condition. The method is based on thin shell theory and the non-rotational flow of non-viscous fluids, in combination with finite element analysis. It is applicable to non-uniform thin anisotropic cylinders subjected to different boundary conditions. The displacement functions for the wall and liquid column are derived from Sanders' equations and from the velocity field associated with the column, respectively. The set of matrices describing their relative contributions to equilibrium is determined by exact analytical integration. The coupled equations are solved for the non-flow problem. For cases of fluid flow, certain analytical modifications are proposed to restore the situation to conventional modal analysis. The non-linear equations of motion are solved by the fourth-order Runge–Kutta numerical method. The frequency variations are then determined with respect to the amplitude of the motion. The trends of the non-linearities are of a softening type.

NOMENCLATURE

(1) *Subscript*

f	fluid
in, out	internal and external flow
(r, x, θ), t	spatial coordinate and time, respectively
s	shell
u	representing internal and external radius, as the case may be
L, NL	linear and non-linear

(2) *Superscript*

D	diagonal matrix
(NL)	non-linear
r	reduced matrix
t	transposed matrix
-1	inversed matrix

(3) *Variables*

a	mean radius of shell
C	speed of sound in the fluid medium
E	Young's modulus
i	$i^2 = -1$
J	number of boundary conditions applied
J_n	Bessel function of the first kind of order n
l	finite element length
m	number of axial half-waves
m_k	defined by relation (10)
N	number of finite elements
n	number of circumferential modes
p, P	fluid pressure on wall
R	average radius of cylindrical shell
R_N	Reynolds number
r	radial coordinate
t	wall thickness
U, V, W	displacements: axial, tangential, radial
U_x	flow velocity
V_r, V_x, V_θ	components of velocity field associated with the flow
x	cylinder generator coordinate
Y_n	Bessel function of the 2nd kind of order n
Z_{pu}	defined by relation (12)

A/t	non-dimensional vibration amplitude
L/r	structural slenderness
α, β	defined by relation (15)
φ	velocity potential
λ_K	complex root of characteristic equation
λ_n	eigenvalue
∇	linear differential operator
ν	Poisson's ratio
ρ_f, ρ_s	density of liquid and solid, respectively
θ	circumferential coordinate
ω_L, ω_{NL}	angular velocity
$x_{ij}, \Gamma_{ij}, \Omega_{ij}$	defined by relation (50)

(4) *Vectors and matrices*

$[m_f]$	inertia matrix for one finite element and defined by relation (16)
$[m_s]$	inertia matrix associated with the shell
$[C_f]_L, [C_f]_{NL}$	damping matrix eqns (17) and (23)
$[K_f]_L, [K_f]_{NL}$	stiffness matrix eqns (18) and (25)
$[K_s]$	stiffness matrix associated with the shell
$[KC_f]$	defined by eqn (24)
$[S_f]^2$	defined by relation (19)
$[D_f]_L, [D_f]_{NL}$	defined by relations (20), (26)
$[G_f]_L, [G_f]_{NL}$	defined by relations (21), (28)
$[GD_f]_{NL}$	defined by relation (27)
$\{\eta\}, \{\Delta\}$	displacement vectors expressed as natural and generalized coordinates, respectively.

1. INTRODUCTION

This study presents a general approach to the non-linear analysis of thin cylindrical anisotropic shells partially or completely filled with liquid under flow or no-flow conditions. The method combines finite element analysis and classical thin shell theory. The finite element chosen was cylindrical (Fig. 1) and was bounded by two circular nodes. There were four degrees of freedom at each node: axial, radial and circumferential displacement, and rotation. The geometry of the finite element made it possible to use Sanders' (1959)

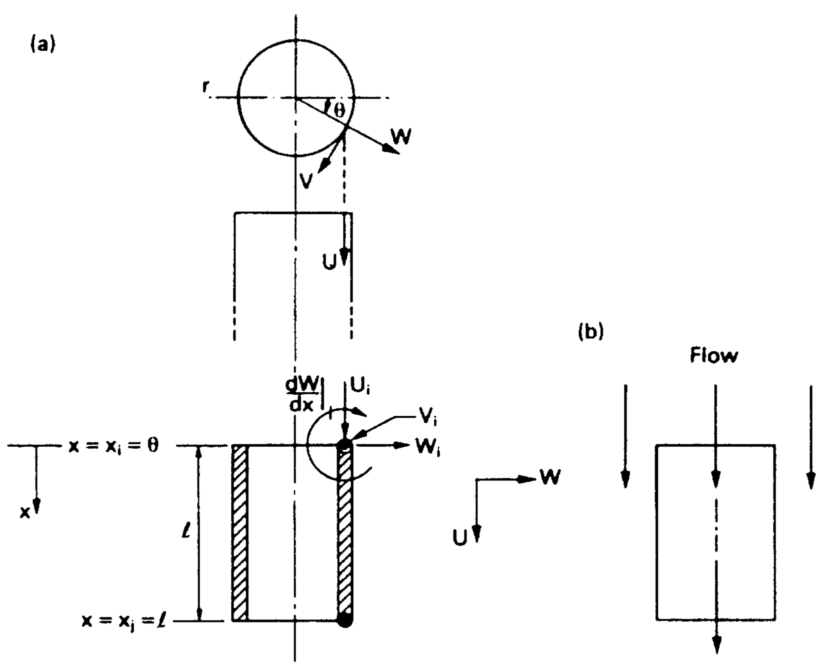

Fig. 1. Displacements and degrees of freedom at a node. (a) Cylindrical finite element *in vacuo*: $m_s\{\ddot{\delta}\}+k_s\{\delta\} = \{0\}$. (b) Cylindrical finite element with flowing fluid (internal and/or external flow): eqn (30).

equations of motion in their entirety to determine the displacement function. This method not only avoids the disadvantages of the Rayleigh–Ritz method, but also satisfies the finite element method convergence criteria (Lakis and Paidoussis, 1972a) and shows greater accuracy than the more usually chosen polynomial functions.

In the present research, we investigated the effect of non-linearities associated with Bernoulli's equation on the natural frequencies of an interactive fluid–shell system. The following experimental parameters were used in the analysis: circumferential mode, structural slenderness ratio, Reynolds number, vibration mode coupling and uncoupling and the effect of composite materials. We considered only the shell's breathing mode (i.e. where the longitudinal axis of the shell remains immobile during structure excitation).

The analytical solution was reached in two stages:

(1) Using the linear strain–displacement and stress–strain relationships which were inserted into Sanders' equation of equilibrium, we determined the displacement functions by solving the linear equation system. We then determined the mass and stiffness matrices for a finite element (Lakis and Paidoussis, 1972a) and assembled the matrices for the complete shell.

(2) From our solution of the velocity potential equation we derived an expression of non-linear pressure as a function of:

(a) the nodal displacement of the fluid element,
(b) the inertial, centrifugal and Coriolis forces,
(c) a combination of non-linear effects.

Through the usual finite element procedure, we obtained the linear mass, damping and stiffness matrices for the fluid (Lakis, 1976b) as well as the non-linear fluid load matrices.

2. ASSUMPTIONS

In order to study the equilibrium of a cylindrical shell including the membrane and bending effects on the reference surface, we used first-order Sanders' (1959) equations. These equations are based on Love's (1944) First Approximation and, unlike other formulations, yield zero deformation for rigid-body motion.

The assumptions for the analysis were as follows:

—The shell is made up of one or more layers of isotropic or orthotropic material.
—Displacements of the wall are sufficiently small to obtain geometric linearity.
—The terms for rotary inertia and shear deformation are neglected.
—Fluid characteristics: non-viscous
incompressible.
—Flow attributes: non-rotational
potential
frictionless.
—The constants for internal pressure and surge pressure are ignored.

3. METHOD

The linear structural and fluid-load matrices were constructed using the procedure described in Lakis (1976b) and Lakis and Paidoussis (1972a). The non-linear fluid-load matrices were determined by development of the second-order Bernoulli equation.

Through modal analysis we transformed our equation of motion according to the axes of the natural coordinates. This analysis varies with the type of vibration encountered. A standard procedure was used for undamped free vibrations. For damped free vibrations, we propose a method in which all information contained in the eigenvalues is considered and post-processing is carried out on the eigenvector matrix.

The coupled equations of motion were solved by means of the method used in Singh *et al.* (1974).

4. MATRIX CONSTRUCTION: NO FLUID

The complete development leading to the construction of the shell's linear structural matrices is given in Lakis (1976b) and Lakis and Paidoussis (1972a).

5. ANALYTICAL FORMULATION: WITH FLUID

5.1. *Dynamic pressure*

Taking a cylindrical shell with a vertical generator axis, we used the procedure outlined in Section 1 within the constraints of the assumptions listed in Section 2. Thus, a suitable potential function is inserted into the Eulerian equations for mass and momentum transport. The appropriate energy expression for the system under consideration is a combination of the first and second laws of thermodynamics, which for isentropic flow along a streamline allows the introduction of the speed of sound for a barotropic fluid.

For ideal, frictionless flow, the velocity potential (Anderson, 1982) is governed by:

$$\nabla^2\varphi = \frac{1}{C^2}[\nabla_\varphi\cdot(\nabla\varphi\cdot\nabla)\nabla\varphi+\varphi_{,tt}+2\nabla\varphi\cdot\nabla\varphi_{,t}] \tag{1}$$

where C is the speed of sound in the fluid medium, φ the velocity potential and

$$(\ )_{,t} = \frac{\partial(\ )}{\partial t}.$$

The development of this third-order non-linear equation in φ is given in Appendix B-1 of Lakis and Laveau (1988).

The linear form of the relation in (1) is expressed by:

$$\nabla^2\varphi = \frac{1}{C^2}[\varphi_{,tt}+2U_x\varphi_{,xt}+U_x^2\varphi_{,xx}]. \tag{2}$$

Furthermore, for steady flow, the velocity potential must satisfy the Laplace Equation. This relation is expressed in the cylindrical coordinate system by:

$$\nabla^2\varphi = \frac{1}{r}(r\varphi_{,r})_{,r}+\frac{\varphi_{,\theta\theta}}{r^2}+\varphi_{,xx}. \tag{3}$$

We define the velocity field associated with this flow by:

$$V_x = U_x+\varphi_{,x}$$
$$V_\theta = \frac{\varphi_{,\theta}}{r}$$
$$V_r = \varphi_{,r} \tag{4}$$

where U_x is the velocity associated with the flow rate by assuming the fluid to be inviscid.

A full definition of the flow requires that two conditions be applied to the shell–fluid interface. The impermeability condition ensures contact between the shell surface and the fluid. This should be:

$$V_r|_{r=a} = \varphi_{,r}|_{r=a} = (W_{,t}+U_xW_{,x})|_{r=a}. \tag{5}$$

Using eqns (1)–(5), we obtain the dynamic pressure as follows:

$$P_u = -\rho_{fu}\left\{\varphi_{,t}+U_{xu}\varphi_{,x}+\frac{1}{2}\left[(\varphi_{,x})^2+\frac{(\varphi_{,\theta})^2}{r^2}+(\varphi_{,r})^2\right]\right\}\Big|_{r=\zeta} \tag{6}$$

where u subscript represents "internal" or "external" as the case may be:

$$\text{if } u = \text{i then } \xi = a_i = a - t/2$$

$$\text{if } u = \text{e then } \xi = a_e = a + t/2.$$

The development for eqn (6) is given in Appendix B-2 of Lakis and Laveau (1988) [see also Lakis (1976b)].

The differential equation is solved using the method of separation of variables. The form of the radial displacement and velocity potential (Lakis, 1976b) is:

$$W_k = C_k \exp\left[i\left(n\theta + \lambda_k \frac{x}{a} + \omega t \right) \right] \tag{7}$$

$$\varphi = \sum_{k=1}^{8} \varphi_k = \sum_{k=1}^{8} R_k(r) S_k(\theta, x, t) \tag{8}$$

where λ_k is the kth root of the characteristic equation and ω the natural angular frequency.

Applying the impermeability condition, we determine S_k (θ, x, t) explicitly. If we set relations (2) and (3) as equal, we will obtain the ordinary homogeneous differential Bessel equation:

$$\frac{r^2\, d^2 R_k(r)}{dr^2} + \frac{r\, dR_k(r)}{dr} + R_k(r)[i^2 m_k^2 r^2 - n^2] = 0 \tag{9}$$

where

$$m_k = \left(\frac{\lambda_k}{a_u}\right)^2 - \frac{1}{c^2}\left\{ w + U_{xu}\frac{\lambda_k}{a_u} \right\}^2. \tag{10}$$

We carry the Bessel equation solution back into (8) to obtain the final expression of the velocity potential evaluated at the cylinder wall:

$$\varphi_u(r, \theta, x, t)_k = a_u Z_{ku}(\mathrm{i}m_k a_u)[W_{k,t} + U_{k,x}] \tag{11}$$

where

$$Z_{ku}(\mathrm{i}m_k a_u) = \frac{1}{n - \mathrm{i}m_k a_u \dfrac{J_{n+1}(\mathrm{i}m_k a_u)}{J_n(\mathrm{i}m_k a_u)}} \quad \text{if} \quad u = \mathrm{i}$$

$$Z_{ku}(\mathrm{i}m_k a_u) = \frac{1}{n - \mathrm{i}m_k a_u \dfrac{Y_{n+1}(\mathrm{i}m_k a_u)}{Y_n(\mathrm{i}m_k a_u)}} \quad \text{if} \quad u = \mathrm{e}. \tag{12}$$

Substituting (11) into the non-linear boundary condition expression (6), we obtain the equation for the pressure on the cylinder wall. It is useful to separate the total pressure into its linear and non-linear terms:

$$P = \{P_{\text{in}} - P_{\text{out}}\}_{\text{L}} + \{P_{\text{in}} - P_{\text{out}}\}_{\text{NL}} \tag{13}$$

where

$$P_{u\text{L}} = -\rho_{fu} \sum_{p=1}^{8} a_u Z_{pu}[W_{p,tt} + 2U_{xu} W_{p,xt} + U_{xu}^2 W_{p,xx}] \tag{14}$$

$$P_{uNL} = -\frac{\rho_{fu}}{2}\sum_{p=1}^{8}\sum_{q=1}^{8}\{\alpha_p\alpha_q[W_{p,xt}W_{q,xt}+2U_{xu}W_{p,xx}W_{q,xt}+U_{xu}^2W_{p,xx}W_{q,xx}] + \beta_p\beta_q[W_{p,t}W_{q,t}+2U_{xu}W_{p,t}W_{q,x}+U_{xu}^2W_{p,x}W_{q,x}]\} \quad (15)$$

where

$$\alpha_p = a_u Z_{pu}$$
$$\beta_p = 1-nZ_{pu}$$
$$\beta_q = 1+nZ_{qu}.$$

5.2. *Linear matrices for the fluid column*

We introduce the nodal interpolation functions for the fluid, which are compatible with the functions for the shell, into the dynamic pressure expression in (14) and execute a series of intermediate matrix operations made necessary by our choice of method. The mass, damping and stiffness matrices for the fluid are obtained by evaluating the following integral (Lakis, 1976b):

$$\int_A [N_F]^t\{P_L\}\,dA.$$

Finally, we have:

$$[m_f] = [A_f^{-1}]^t[S_f][A_f^{-1}] \quad (16)$$

$$[c_f] = [A_f^{-1}]^t[D_f][A_f^{-1}] \quad (17)$$

$$[k_f] = [A_f^{-1}]^t[G_f][A_f^{-1}] \quad (18)$$

where

$$S_f(p,q) = \pi[-\delta_i\gamma_i^2 Z_q I_{pqi} + \delta_e\gamma_e^2 Z_{qe} I_{pqe}] \quad (19)$$

$$D_f(p,q) = 2i\lambda_q\pi[-\delta_i\gamma_i\bar{U}_i Z_{qi} I_{pqi} + \delta_e\gamma_e\bar{U}_e Z_{qe} I_{pqe}] \quad (20)$$

$$G_f(p,q) = -\lambda_q^2\pi[-\delta_i\bar{U}_i^2 Z_{qi} I_{pqi} + \delta_e\bar{U}_e^2 Z_{qe} I_{pqe}] \quad (21)$$

and $p, q = 1, \ldots, 8$.

In eqns (19)–(21) we define the following non-dimensional quantities:

$$\delta_u = \frac{a_u\rho_{fu}}{t_1\rho_1} \qquad U_0^2 = \frac{p(1,1,1)}{\rho_1 t_1}$$
$$\bar{U}_u = \frac{U_{xu}}{U_0} \qquad \gamma_u = \frac{a_u}{r_1}$$

where ρ_1 is the density, t the thickness of the first finite element of the shell, r the radius and $p(1,1,1)$ the first element of **[P]**

$$I_{pqu} = \frac{\exp\left[i(\lambda_p+\lambda_q)\frac{l_j}{a_u}\right]-1}{i(\lambda_p+\lambda_q)} \quad \text{if} \quad \lambda_p+\lambda_q \neq 0$$

$$I_{pqu} = \frac{l_j - l_i}{a_u} \quad \text{if} \quad \lambda_p+\lambda_q = 0. \quad (22)$$

5.3. *Development of the non-linear fluid-load matrices*

We use the procedure outlined in the previous section, ignoring the cross products in the non-linear dynamic pressure expression (15). We obtain the following matrices for the non-linear effects:

$$[c_{f_{NL}}] = [A_f^{-1}]^t[D_f][A_f^{-1}] \tag{23}$$

$$[kc_{f_{NL}}] = [A_f^{-1}]^t[GD_f]_{NL}[A_f^{-1}] \tag{24}$$

$$[k_{f_{NL}}] = [A_f^{-1}]^t[G_f]_{NL}[A_f^{-1}] \tag{25}$$

where:

$$D_{f_{NL}}(p,q) = \frac{e^{6\pi i n}-1}{6in}\{\delta_i\gamma_i J_{pqi}[Z_{qi}^2[n^2+\lambda_q^2]-1]-\delta_e\gamma_e J_{pqe}[Z_{qe}^2[n^2+\lambda_q^2]-1]\} \tag{26}$$

$$GD_{f_{NL}}(p,q) = \frac{(e^{6\pi i n}-1)\lambda_q i}{3in}\{\delta_i\bar{U}_i J_{pqi}[Z_{qi}^2[n^2+\lambda_q^2]-1]-\delta_e U_e J_{pqe}[Z_{qe}^2[n^2+\lambda_q^2]-1]\} \tag{27}$$

$$G_{f_{NL}}(p,q) = \frac{-(e^{6\pi i n}-1)\lambda_q^2}{6in}\left\{\frac{\delta_i\bar{U}_i^2}{\gamma_i}J_{pqi}[Z_{qi}^2[n^2+\lambda_q^2]-1]-\frac{\delta_e\bar{U}_e^2}{\gamma_e}J_{pqe}[Z_{qe}^2[n^2+\lambda_q^2]-1]\right\} \tag{28}$$

where:

$$J_{pqu} = \frac{\exp\left[i(\lambda_p+2\lambda_q)\frac{l_i}{a_u}\right]-1}{i(\lambda_p+2\lambda_q)} \quad \text{if} \quad \lambda_p+2\lambda_q = 0$$

$$J_{pqu} = \frac{l_j-l_i}{a_u} \quad \text{if} \quad \lambda_p+2\lambda_q = 0. \tag{29}$$

6. ANALYSIS OF FREE VIBRATIONS

6.1. *Global matrices*

The motion of a shell element interacting with a fluid column is governed by the equations of motion in generalized coordinates:

$$[[m_s]-[m_f]]_L\{\ddot{\delta}\}-[c_f]_L\{\dot{\delta}\}+[[k_s]-[k_f]]_L\{\delta\}-[c_f]_{NL}\{\dot{\delta}^2\} - [kc_f]_{NL}\{\delta\dot{\delta}\}-[k_f]_{NL}\{\delta\}^2 = \{0\} \tag{30}$$

where subscripts s and f refer to the shell *in vacuo* and fluid-filled, respectively; $\{\delta\}$ is the degrees of freedom vector for the total nodes. m_s and k_s are, respectively, the mass and stiffness matrices of the shell *in vacuo* and they are developed in Lakis and Paidoussis (1972a). The total structure motion is governed by an analogous equation which we shall write as:

$$[[M_S]-[M_F]]_L\{\ddot{\Delta}\}-[C_F]_L\{\dot{\Delta}\}+[[K_S]-[K_F]]_L\{\Delta\}-[C_F]_{NL}\{\dot{\Delta}^2\} - [KC_F]_{NL}\{\Delta\dot{\Delta}\}-[K_F]_{NL}\{\Delta^2\} = \{0\}. \tag{31}$$

These global matrices are obtained by assembling the element matrices. After the boundary conditions are applied, these matrices are reduced to square matrices of order $4(N+1)-J$ where J is the number of restrictions imposed.

To abbreviate the expression, we set:

$$[M_T] = [[M_S]-[M_F]]_L \quad \text{and} \quad [K_T] = [[K_S]-[K_F]]_L$$

$$[M_T^r]\{\ddot{\Delta}\} - [C_L^r]_F\{\dot{\Delta}\} + [K_T^r]\{\Delta\} - [C_F^r]_{NL}\{\Delta^2\} - [KC_F^r]_{NL}\{\dot{\Delta}\Delta\} - [K_F^r]_{NL}\{\Delta^2\} = \{0\} \tag{32}$$

where r means reduced.

Let us set

$$\{\Delta^r\} = [\varphi]\{\eta\} \tag{33}$$

where $[\varphi]$ is the square eigenvector matrix in the symmetric linear matrix system, $\{\eta\}$, $\{\Delta^r\}$ the displacement vectors expressed as natural and generalized coordinates, respectively, and [^{r}] the ... matrix ... generalized coordinates.

We first substitute expression (33) into (32), then multiply (32) by $[\varphi]^t$ to obtain, finally:

$$[M_T^D]\{\ddot{\eta}\} - [C_L^D]\{\dot{\eta}\} + [K_T^D]\{\eta\} - [\varphi^t][C_{NL}][\varphi^2]\{\dot{\eta}^2\}$$
$$- [\varphi^t][KC_{NL}][\varphi^2]\{\dot{\eta}\eta\} - [\varphi^t][K_{NL}][\varphi^2]\{\eta^2\} = \{0\} \tag{34}$$

where

$$[M^D] = [\varphi]^t[M_T][\varphi]$$
$$[C^D] = [\varphi]^t[C_L][\varphi]$$
$$[K^D] = [\varphi]^t[K_T][\varphi]$$

where D stands for diagonal

[^{D}] the ... matrix ... natural coordinates.

The matrices quantifying the fluid column contribution to the matrix equations of motion are non-symmetric. To facilitate the analysis, therefore, we consider only the symmetric portion of the matrices. We will see later that this simplification is justified.

6.2. *No-flow condition*

Under stagnant conditions, eqn (31) reduces to:

$$[M]_T\{\ddot{\Delta}\} + [K]\{\Delta\} - [C]_F\{\dot{\Delta}^2\} = \{0\}. \tag{35}$$

First we solve for the linear case to obtain the $4*(N+1)-J$ eigenvalues and eigenvectors. With this eigenvector matrix, we then develop the coupled equations of motion in natural coordinates.

6.3. *Flow conditions*

As mentioned in Section 3, we make some modifications to the procedure described above.

Due to the presence of a non-proportional damper, we reduce the second-order linear system to a first-order system. We again consider only the symmetric portions of the fluid matrices, as given by the equation:

$$[M]\{\ddot{q}\} + [C]\{\dot{q}\} + [K]\{q\} = \{0\} \tag{36}$$

which may be represented in the following form (Meirovitch, 1967):

$$[A]\{\dot{y}\}+[B]\{y\} = \{0\} \tag{37}$$

where:

$$[A] = \begin{bmatrix} 0 & [M] \\ [M] & [C] \end{bmatrix}$$

$$[B] = \begin{bmatrix} -[M] & 0 \\ 0 & [K] \end{bmatrix}$$

$$\{y\} = \begin{bmatrix} \{\dot{q}\} \\ \{q\} \end{bmatrix}.$$

We assume the form of the homogeneous solution to be:

$$\{y(t)\} = e^{\lambda t}\{\psi\}. \tag{38}$$

Equation (36) becomes:

$$\lambda[A]\{\psi\} = -[B]\{\psi\}. \tag{39}$$

From the solution of (39), we obtain a series of $2n$ eigenvalues and $2n$ eigenvectors. For each given eigenvalue, λ_n, the corresponding eigenvectors are developed as follows:

$$\{\psi_n\} = \begin{bmatrix} \lambda\{\varphi_n\} \\ \{\varphi_n\} \end{bmatrix} \quad n = 1,2,\ldots,N \tag{40}$$

where N is the number of degrees of freedom.

The eigenvalues are complex and always occur in conjugate pairs.

The dimensional incompatibility between the eigenvector matrices defined in (33) and (40) and the absence of any weighted orthogonal relationships between the $[M]$, $[C]$ and $[K]$ matrices in (36) are difficulties which have to be overcome.

We attempt to determine the $m_{ii}\cdot c_{ii}$ and k_{ii} for each of the uncoupled equations of motion. To this end we have two pieces of information on the state of the damped system for each eigenvalue pair.

We have:

$$\lambda_n = \alpha+i\beta. \tag{41}$$

According to (38) and (41), we also have:

$$y_n(t) = \psi_n e^{\lambda_n t} = \psi_n e^{\alpha_n t} e^{i\beta_n t}. \tag{42}$$

Now, for a damped system, the displacement is defined in Meirovitch (1967):

$$y_n(t) = Y_0 e^{-\xi\omega_n t} e^{i\omega_d t} \tag{43}$$

where Y_0 is the initial amplitude, ξ the critical damping coefficient, ω_n the natural angular frequency and ω_d the damped angular frequency.

We then, by analogy, associate the terms in eqns (42) and (43) to find our unknowns. Finally, we obtain:

$$k_{nn} = (\alpha_n^2+\beta_n^2)m_{nn} \tag{44}$$

$$c_{nn} = 2\alpha_n m_{nn}. \tag{45}$$

The m_{nn} are taken from the orthogonal relation:

$$\{\psi\}^t[B]\{\psi\} = 0. \tag{46}$$

The first n diagonal terms of the resulting matrix are the n elements of diagonal $[M]$.

The eigenvector matrix is constructed by selecting and assembling side-by-side the n column vectors $\{\varphi_N\}$ in (40) associated with each pair of eigenvalues λ_N. With this $[\varphi]$, we then go on to couple the equations of motion in natural coordinates (34).

6.4. *Solving the coupled equations*

During the modal analysis, we have to consider coupling between the different modes since products of the form $[\varphi]^t\,[M]\,[\varphi]^2$ do not generally give diagonal matrices.

A typical system equation, therefore, would be of the form:

$$m_{ii}\ddot{\eta}_i - c_{ii}\dot{\eta}_i + k_{ii}\eta_i - \sum_{j=1}^{\text{NREDUC}} \{C_{ij}\dot{\eta}_i\dot{\eta}_j + KC_{ij}\eta_i\dot{\eta}_j + K_{ij}\eta_i\eta_j\} = 0. \tag{47}$$

Let us set:

$$\eta_i(\tau) = A_i f_i(\tau) \tag{48}$$

which satisfies the initial conditions

$$f_i(0) = 1 \quad \text{and} \quad \dot{f}_i(0) = 0 \tag{49}$$

where A_i is the vibration amplitude.

Equation (47), after simplifying by A_i and dividing by m_{ii}, becomes:

$$\ddot{f}_i - \frac{1}{\tau_i}\dot{f}_i + \omega_i^2 f_i - \sum_{j}^{\text{NREDUC}} \left(\frac{A_i}{t}\right)\{\Gamma_{ij}\dot{f}_i\dot{f}_j + \chi_{ij}f_i\dot{f}_j + \Omega_{ij}f_i f_j\} = 0 \tag{50}$$

where

$$\omega_i^2 = \frac{k_{ii}}{m_{ii}}, \quad \frac{1}{\tau_i} = \frac{c_{ii}}{m_{ii}}, \quad \Gamma_{ij} = \frac{C_{ij}^{(\text{NL})}t}{m_{ii}},$$

$$\chi_{ij} = \frac{KC_{ij}^{(\text{NL})}t}{m_{ii}}, \quad \Omega_{ij} = \frac{K_{ij}^{(\text{NL})}t}{m_{ii}}$$

where t is the shell thickness.

If, however, we ignore non-linear coupling between the natural coordinates, eqn (50) will take the form:

$$\ddot{f}_i - \frac{1}{\tau_i}\dot{f}_i + \omega_i^2 f_i - \left(\frac{A_i}{t}\right)\{\Gamma_{ii}\dot{f}_i^2 + \chi_{ii}f_i\dot{f}_i + \Omega_{ii}f_i^2\} = 0. \tag{51}$$

The solution $f_i(\tau)$ of these ordinary non-linear differential equations which satisfies the initial conditions (49) is approximated numerically by a fourth-order Runge–Kutta calculation. The linear and non-linear natural angular frequencies are evaluated by a systematic search for the $f_i(\tau)$ roots as a function of time. The $\omega_{\text{NL}}/\omega_{\text{L}}$ ratio is expressed as a function of non-dimensional ratio $A_i t$.

7. CALCULATIONS AND DISCUSSION

In this section, we discuss the application of the proposed new method to a number of cases. First, calculations were performed to demonstrate the validity of the simplifying hypotheses we described in (50). Secondly, we conducted a systematic investigation into the influence of the non-linearities associated with Bernoulli's equation by considering the experimental parameters listed in Section 2. In conclusion, we briefly discuss the stability problem inherent in the dynamic behaviour of the equation system studied.

Development of the analytical model required an additional hypothesis. Indeed, with the present state of our knowledge, we are unable to apply the orthogonality properties of the modal vectors to a general eigenvalue problem (Meirovitch, 1967; Wilkinson, 1965). We are thus simplifying the parameters and limiting our dynamic analysis strictly to consideration of those shell–fluid systems which lead to a symmetric matrix system of eigenvalues.

This simplifying hypothesis is validated if the resultant eigenvalues come close to the original system. Tables 1–4 of Lakis and Laveau (1988) show the variance between the eigenvalues in the original and simplified systems, corresponding to cases of damped and undamped free vibration. A trend was observed toward minimum variance at the extreme modes and maximum variances (15% and 19.5% for a damped and an undamped case, respectively) at the median modes. The simplification therefore seems to be valid since it shows the two systems to have comparable dynamic behaviour.

We used four cylindrical shell models to investigate the influence of a fluid medium. These models are presented in Table 1.

The shell had the properties given in Obraztsova and Shklyarchuk (1979)

$$E = 20685.0\ \text{MPa} \qquad \text{(steel)}$$
$$\nu = 0.29$$
$$R = 193.75\ \text{mm} \qquad R/t = 150.0$$
$$t = 1.29117\ \text{mm}$$
$$\rho_{\text{shell}} = 7812.5\ \text{kg m}^{-3} \qquad \frac{\rho_{\text{water}}}{\rho_{\text{shell}}} = 0.128.$$
$$\rho_{\text{water}} = 1000.0\ \text{kg m}^{-3}$$

The boundary conditions were for a shell simply-supported at both ends, such that $v = w = 0$.

The parameters of the investigation took the following values:

—$n = 3$ and 6 corresponding to the third and sixth circumferential vibration modes
—structural slenderness ratio $L/R = 3$ and 6
—Reynolds number, R_N
where

$$R_N = \frac{2U_x R\rho}{\nu}$$

in which U_x is the mean velocity of the external flow relative to the motion of the solid, R is the average radius of the cylindrical shell and ρ and ν are respectively the density and viscosity of the fluid flowing therein
$R_N = 0$ (no-flow condition)
$R_N = 1.0\text{E}+06$ and $2.0\text{E}+06$.

The structural damping is neglected because it is much lower than the fluid damping. For the no-flow condition, the dynamic behaviour is essentially undamped vibration. When flow takes place, the fluid damping is dominant.

All the graphs illustrating our analysis may be found in Appendix C of Lakis and Laveau (1988). A synopsis of their overall characteristics is given in Table 1.

Table 1. Synoptic table. Influence of different experimental parameters on their relative contribution to variations in the frequency ratio. Natural angular frequency f_{NL}/f_L

Model					Axial mode coupling			Axial mode uncoupling		
	Experimental parameters				Range of values			Range of values		
No.	n	$\frac{L}{R}$	Young's modulus	Reynolds number R_N	$\frac{f_{NL}}{f_L}$	$\frac{A}{t}$	3rd axial mode	$\frac{f_{NL}}{f_L}$	$\frac{A}{t}$	3rd axial mode
1	3	6	E_{steel}	0×10^0	0.875@1.000	10^8@10^{14}	10-12-9	0.800@1.000	10^9@10^{18}	9-7-10
				1×10^6	0.990@1.000	10^{13}@10^{23}	1-3-2	0.935@1.000	10^{14}@10^{27}	1-3-7
				2×10^6	0.997@1.000	10^{13}@10^{23}	1-3-2	0.950@1.000	10^{13}@10^{26}	1-3-7
2	6	6	E_{steel}	0×10^0	0.935@1.000	10^8@10^{14}	11-10-12	0.800@1.000	10^9@10^{17}	12-10-11
				1×10^6	0.992@1.000	10^{17}@10^{22}	1-3-7	0.925@1.000	10^{17}@10^{26}	1-3-7
3	3	3	E_{steel}	0×10^0	0.900@1.000	10^{12}@10^{17}	12-9-7	0.800@1.000	10^{12}@10^{21}	12-10-7
				1×10^6	0.990@1.000	10^{23}@10^{31}	2-9-11	0.910@1.000	10^{24}@10^{31}	2-11-10
4	3	6	$E_{steel}/100$	0×10^0	0.900@1.000	10^8@10^{14}	10-12-9	0.800@1.000	10^9@10^{18}	9-7-10
				1×10^6	0.998@1.000	10^8@10^{18}	1-3-2	0.900@1.000	10^9@10^{23}	1-3-2

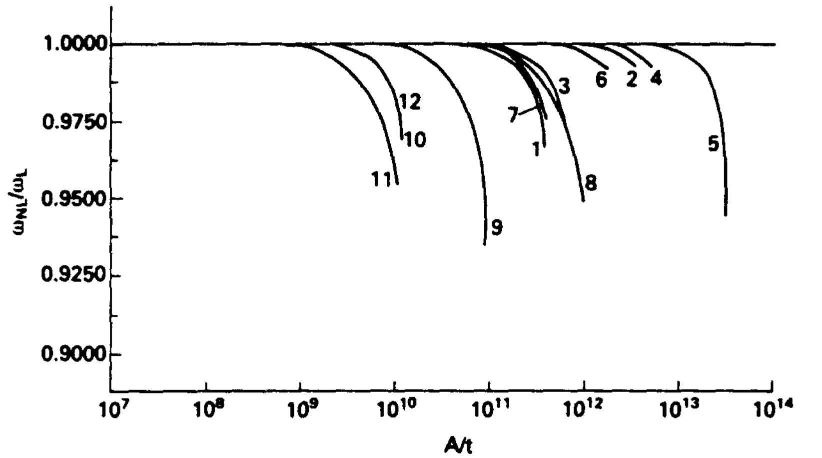

Fig. 2. Variations in frequency ratio as a function of motion amplitude. Model No. 2, $R_N = 0.0$, axial mode coupling.

It can be seen in Figs 2–5 that the variation in frequency ratio (ω_{NL}/ω_L) decreases with increases in the non-dimensional amplitude ratio. There does, therefore, appear to be a generalized non-linear trend of the softening type. In addition, the earliest occurrences of sensitivity to this non-linearity are associated with a clustering of the first and last axial modes for cases of damped and undamped vibration, respectively.

We shall now describe the parameter variables in terms of their relative impact on frequency ratio variations. In all cases, uncoupling the equations of motion produces more pronounced variation than coupling. Therefore the solution of the coupled equations provides a better assessment of the relative weight of the effect analyzed. The variation is low to moderate for no-flow and moderate to high for flowing fluid columns. In conclusion, the following phenomena, in descending order of significance, had an impact on the other parameters of our study: the circumferential vibration mode, composite material effect and structural slenderness ratio.

A glance at the synopsis in Table 1 reveals the large amplitude required to obtain any tangible effect. An amplitude of this magnitude corresponds to a pump discharge pulse or "water hammer" effect!

It would be of interest nevertheless to explore the possible influence of the rigidity ($E = E_{steel}/100$) further, since the order of magnitude for the non-dimensional amplitude

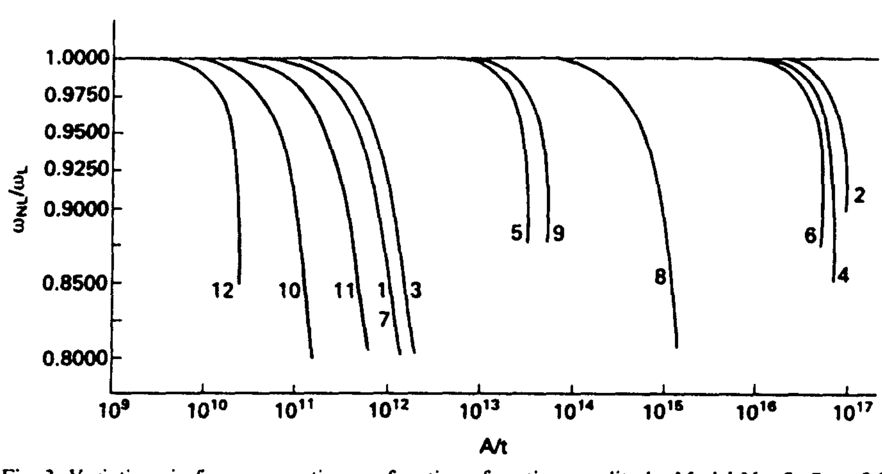

Fig. 3. Variations in frequency ratio as a function of motion amplitude. Model No. 2, $R_N = 0.0$, axial mode uncoupling.

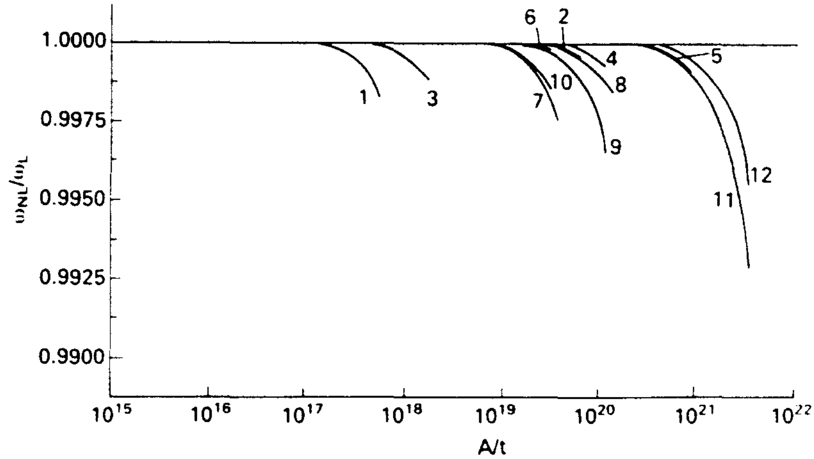

Fig. 4. Variations in frequency ratio as a function of motion amplitude. Model No. 2, $R_V = 1.0 \times 10^6$, axial mode coupling.

ratio for that is the closest to unity. There are two other avenues of investigation which hold out inviting prospects: the boundary condition and compressibility effects. The jet engine nozzle, which is a clamped-free structure, is a particular structure that falls perfectly into our analytical model. It is possible for coupling to occur between the modes associated with floating instabilities, and the non-linearity can be studied when the Mach number tends toward 0.30.

Our investigation to date has thus confirmed the hypothesis that the influence of Bernoulli equation non-linearity on the dynamic behaviour of the shell–fluid structure is negligible.

We had some difficulty in stabilizing the solution of the non-linear equations of motion under one particular condition. This condition related to the magnitude of the contribution by the non-linear effects to system inertia. Furthermore, damped structures present greater numerical stability than undamped structures.

The instabilities were characterized either by oscillations of which the frequencies varied substantially through time, or by an inconsistency between the frequency and rate of these oscillations around the balanced position. This occurred right from the initial pulse.

We adopted a convergence criterion which would keep us within the realm of stability. The criterion involved ensuring that the solution would not repeat by considering a frequency response acceptable if it fell within a band under 1% of its nominal value.

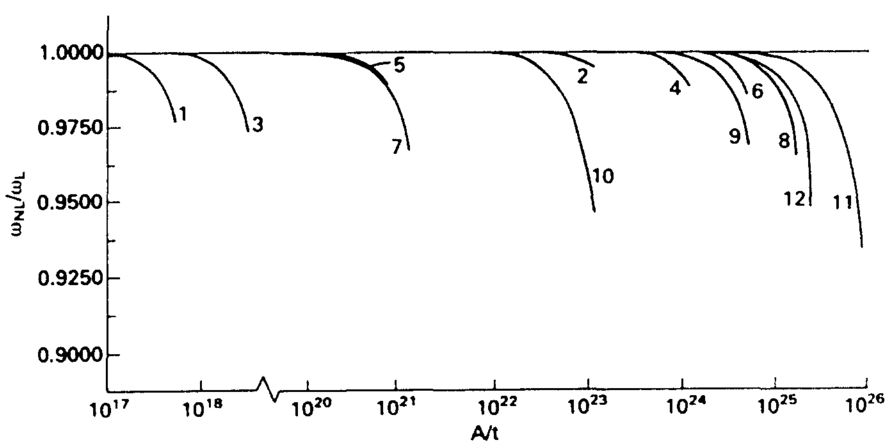

Fig. 5. Variations in frequency ratio as a function of motion amplitude. Model No. 2, $R_V = 1.0 \times 10^6$, axial mode uncoupling.

8. CONCLUSIONS

We developed a method based on Sanders' thin-shell equation, the equation for non-rotational and frictionless fluid flow and on finite element analysis. The method predicts the influence of non-linearity in association with flow definition on the dynamic behaviour of vertical cylindrical shells partially or completely filled with a stagnant or flowing liquid. Our model does not yet predict the dynamic behaviour of vessels partially filled with a stagnant liquid. For this reason, we expect to incorporate the free shear boundary condition and the sloshing effects in our next analysis.

The finite element was cylindrical and geometrically axisymmetric. The displacement functions were therefore derived from the equations of motion for the shell. The mass and stiffness matrices were determined by exact analytical integration. The displacement functions for the fluid column were derived from the velocity field associated with the column and from the non-linear impermeability and dynamic conditions applied to the shell–fluid interface. The matrices for the fluid column contribution were determined in a manner similar to the matrices for the shell wall. Conventional modal analysis was used to treat a shell with undamped free vibrations and vibrations damped with a non-proportional damper. This latter vibration case required both a simplifying hypothesis, which proved to have been justified, and a few analytical modifications. The non-linear equations of motion expressed in natural coordinates were solved using a numerical time-based integral method: the fourth-order Runge–Kutta numerical method.

This area of investigation is still wide open and there is very little on the subject in the literature. We are unable, therefore, to confirm whether, in the context of a dynamic analysis, we are justified in completely neglecting the influence of the non-linear boundary condition at the shell–fluid interface. The present theory was formulated and applied to straight cylindrical shells with circular sections. It can, however, be used to analyze a shell of revolution with arbitrary curvature by appropriate assembly of the cylindrical, conical or spherical elements to approximate the geometry desired. It would be interesting to apply the method developed to investigations of forced vibrations in a cylindrical shell subjected to dynamic loads. It would also be interesting to include phenomena of flotation and buckling instability in this analysis. In conclusion, the next logical step in the work of our group would be the investigation of the effect of geometric non-linearities of the walls on the dynamic behaviour of shell–fluid interaction.

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